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# Exact solution of the totally asymmetric Oslo model

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## Abstract

Recently it has been found (Pruessner G and Jensen H J 2003 *Phys. Rev. Lett.* **91** 244303) that a totally asymmetric variant of the Oslo model (Christensen K *et al* 1996 *Phys. Rev. Lett.* **77** 107) represents the entire universality class of the Oslo model with anisotropy. The totally asymmetric model can be solved without scaling assumptions by finding recursively the eigenvectors of the Markov matrix, which can be suitably modified to produce the moment generating function of the relevant observable. This method should be applicable to many other stochastic processes.

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## 1. Introduction

Self-organized criticality (SOC) was originally introduced [1] as an approach to understand  $1/f$ -noise as well as the apparent abundance of power laws in nature, which is generally accepted as the sign of scale invariance. The idea is that under very general circumstances driven stochastic processes develop into a scale-invariant state without the explicit tuning of parameters, contrary to what one would expect from equilibrium critical phenomena [2].

A very large zoo of SOC models has been developed [3], with each model having certain special features. However, based on large-scale numerical simulations it has become increasingly clear that many of the models, formerly thought of as representatives of entire universality classes or even paradigms for a specific type of model, are either not scale invariant or at least do not follow simple scaling [4–7]. In fact, models of SOC are notorious for slow convergence and deviations from the expected behaviour.

Some models, however, show all the features one would expect from a ‘self-organized critical’ system: consistent exponents and scaling, universality, crossover between different classes etc. One of these models is the so-called Oslo model [8], which was motivated by an experiment [9]. In a recent paper [10], it has been shown that any (small) amount of anisotropy will drive this model eventually (in the thermodynamic limit) towards another ‘fixed point’,

which is represented by the ‘totally asymmetric Oslo model’ (TAOM). This model is solvable directly on the lattice without making any scaling assumptions. Consequently, it is not only possible to derive exponents, but also to calculate amplitudes of the moments of the relevant observable.

The TAOM is totally asymmetric in the sense that particles can move in one direction only, similar to the totally asymmetric exclusion process (TASEP) [11, 12]. The TASEP has been solved using a matrix product state ansatz [12, 13], so that it seems reasonable to apply similar techniques to the present model. However, there is a crucial difference between these two stochastic processes: the relevant observables in the TASEP exist on a microscopic timescale, i.e. there is an intrinsic timescale in the time evolution of the microstate of the system. In contrast, in the TAOM the relevant observables are obtained by any dynamics which comply with a certain set of rules. In that sense, the specific (microscopic) dynamics of the TAOM are irrelevant. This is reflected in its theoretical treatment, in that the TASEP is updated homogeneously (all sites evolve equally) but the TAOM is perturbed once and is only observed after it is fully relaxed (separation of timescales).

In the following, the model is defined in terms of rules on a lattice. Using a Markov matrix approach it is then solved and exponents and amplitudes derived. After mapping it to a reaction–diffusion process as well as various other processes, a more accessible continuum theory is described.

## 2. The model

The model is defined on a one-dimensional lattice of size  $L$ , where each site  $i = 1, 2, \dots, L$  has assigned a slope  $z_i \in \{1, 2\}$  and a critical slope  $z_i^c \in \{1, 2\}$ . From a flat initial configuration  $z_i \equiv 1$  and  $z_i^c$  random, where  $z_i^c = 1$  is chosen with probability  $p$  and  $z_i^c = 2$  otherwise, the model evolves according to the following rules:

- (i) (Driving) Increase  $z_1$  by one unit (‘initial kick’).
- (ii) (Toppling) If there is an  $i$  where  $z_i > z_i^c$ , decrease  $z_i$  by 1 unit,  $z_i \rightarrow z_i - 1$ , and increase the right nearest neighbour  $j = i + 1$  by 1,  $z_j \rightarrow z_j + 1$  (charging). A new  $z_i^c$  is chosen at random from  $\{1, 2\}$ , where  $z_i^c = 1$  is chosen with probability  $p$  and  $z_i^c = 2$  otherwise.
- (iii) Repeat the second step until  $z_i \leq z_i^c$  (‘stable’) everywhere. Then proceed with the first step.

During toppling, the right neighbour is charged of course only if it actually exists, i.e.  $j \leq L$ , otherwise the toppling site  $i$  relaxes without charging another site, so that a unit leaves the system. Apart from this boundary condition, the TAOM differs from the original Oslo model [8] in redistributing only a unit to the right, rather than one to each side.

It is important to note that the value of  $z_i^c$  is determined only *after* a site has discharged. Thus, if a *stable* site  $i$  is in state  $z_i = 1$ , its value of  $z_i^c$  could be randomly chosen in the moment when it is needed, i.e. when the site is charged again. If a *stable* site  $i$  is in state  $z_i = 2$ , then  $z_i^c$  has necessarily the value 2. When this site is charged, it will relax to  $z_i = 2$  again and a new random  $z_i^c$  is drawn. If that is  $z_i^c = 1$ , then the site topples again and ends up in state  $z_i = 1$ , otherwise it remains in state  $z_i = 2$ .

If all sites are stable, i.e.  $z_i \leq z_i^c$  for all  $i \in [1, L]$ , a configuration is fully described by the values of the  $z_i$  alone; if  $z_i = 2$  then  $z_i^c = 2$ , otherwise  $z_i^c$  is random and has not yet been used in the dynamics.

The number of times the second rule is applied, that is the number of topplings, is the avalanche size  $s$ . The fundamental observable one is interested in is the probability density

function of these sizes,  $P(s)$ , which is expected to obey simple scaling above a fixed lower cutoff  $s_l$ ,

$$P(s) = as^{-\tau} \mathcal{G}(s/s_0) \tag{1}$$

with  $s_0 = bL^D$ ,  $\mathcal{G}$  the universal finite-size scaling function, and metric factors  $a$  and  $b$  [14]. Equation (1) is the *definition* of the two exponents  $\tau$  and  $D$ . It entails that the moments  $\langle s^n \rangle$  of  $P(s)$  behave like [10]

$$\langle s^n \rangle = a(bL^D)^{1+n-\tau} g_n \quad \text{for } 1+n-\tau > 0 \tag{2}$$

with universal amplitudes  $g_n$  [10]. Thus, assuming (1) one can derive  $\tau$  and  $D$  from the behaviour of any two moments. Below, the exponents  $\gamma_n$  from  $\langle s^n \rangle \propto L^{\gamma_n}$  will be used; equation (2) therefore means

$$\gamma_n = D(1+n-\tau). \tag{3}$$

The time series of avalanches,  $s(t)$ , itself is not Markovian, while the sequence of stable configurations of the system, given by the vector  $(z_1, z_2, \dots, z_L)$ , is. Since two consecutive stable configurations are not necessarily linked by a unique sequence of topplings, the sequence of avalanche sizes is not uniquely determined by the sequence of configurations. Nevertheless, in the form of a generating function this ambiguity can be built into the Markov matrix operating on the distribution vector of configurations, so that the avalanche size distribution can be determined by means of this specially prepared Markov matrix.

### 2.1. Abelian property

Put simply, if a model is Abelian [15], it means that the order of updates is irrelevant for its statistical properties. It is exceptionally simple to see this property here: firstly, for the final state of an individual site there is no difference between a certain number of charges arriving at once or arriving sequentially. Secondly, if a site topples, it pours particles on its right neighbour, but it will never receive anything back from the neighbour. So, if a site at  $z = 1$  has received 3 units, it topples at least twice, but for this site it does not make any difference whether it first moves 1 unit over to the right neighbour and waits until all sites to its right have relaxed, or whether it moves all units at once, 2 with probability  $q \equiv 1 - p$  (namely the probability to have  $z_i^c = 2$  after the second toppling) and 3 with probability  $p$ .

In this informal sense, the Abelian property allows the updating to run from left to right, completely relaxing each site during a sweep. If there is no toppling on site  $i$ , the avalanche has stopped and sites  $j > i$  do not need to be checked for the toppling condition  $z_j > z_j^c$  at all. This procedure makes very efficient Monte Carlo simulations possible. Moreover, it defines an activity  $a_i$ , which is the total number of charges received at site  $i$  during an avalanche. The activity will be used in section 4.

### 3. Markov matrix approach

The tensor product  $\otimes$  used here is explained in detail in [16]. In particular it has the property (provided that  $A, B, A'$  and  $B'$  have appropriate ranks)

$$(A \otimes B) \odot (A' \otimes B') = (A \odot A') \otimes (B \odot B') \tag{4}$$

where  $\odot$  stands for the appropriate operator: it is a matrix multiplication if  $A, B, A'$  and  $B'$  are matrices, it is a multiplication of a matrix and a vector if  $A$  and  $B$  are matrices and  $A'$  and  $B'$  are vectors or vice versa, or it is an inner product if they are all vectors. In particular, in the latter case it is

$$(a \otimes b)(a' \otimes b') = (aa')(bb'). \tag{5}$$

First, we consider a single-site system, which can be in exactly two states, so that its distribution of states can be represented by a two-row vector. By convention, the upper row corresponds to  $z = 1$  and the lower row to  $z = 2$ . Three matrices are introduced, corresponding to the three possible outcomes of a single initial kick.

The matrix  $S$  corresponds to a unit being absorbed, i.e. the site is in state  $z_1 = 1$  and  $z_1^c = 2$ , which occurs with probability  $q$ . After the charge, the system is in state  $z_i = 2$ . Similarly,  $T$  corresponds to a single toppling due to the charge and  $U$  corresponds to a double toppling:

$$S = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} \quad T = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad U = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}. \quad (6)$$

In the following, the aim is to find an expression for the moment generating function of the avalanche size distribution. To this end, each matrix is multiplied by an appropriate power of  $x$ , so that evaluating at  $x = 1$  gives the usual Markov matrix of this process, and deriving by  $x$  before evaluating at  $x = 1$  multiplies each process by the number of topplings occurring in it, and similarly for higher order moments [17].

It will be motivated only *a posteriori* that a dissipation process is required, say with probability  $0 \leq \epsilon \leq 1$ ; this process corresponds to charging without changing the state, i.e. an identity operation  $\mathbf{1}$ , the latter being the  $2 \times 2$  identity matrix. The resulting single-site operator is therefore

$$\mathbf{O}_1(x) = \epsilon \mathbf{1} + \delta(S + xT + x^2U) = \begin{pmatrix} \epsilon + x\delta p & x^2\delta p \\ \delta q & \epsilon + x\delta q \end{pmatrix} \quad (7)$$

with  $\delta \equiv 1 - \epsilon$ . The eigenvectors and eigenvalues of this matrix are found to be

$$\begin{aligned} \langle e_\lambda(x) | &= \left( \frac{1}{x}, 1 \right) & |e_\lambda(x)\rangle &= \begin{pmatrix} xp \\ q \end{pmatrix} & \lambda &= \epsilon + x\delta \\ \langle e_\mu(x) | &= \left( -\frac{q}{x}, p \right) & |e_\mu(x)\rangle &= \begin{pmatrix} -x \\ 1 \end{pmatrix} & \mu &= \epsilon \end{aligned} \quad (8)$$

where  $\mathbf{O}_1$  acts on bra-vectors  $\langle |$  from the right and on ket-vectors  $| \rangle$  from the left. The vectors are normalized such that

$$\langle e_a | e_b \rangle = \delta_{a,b} \quad (9)$$

with  $\delta_{a,b}$  denoting the Kronecker delta function. In order to distinguish vectors of different size, in the following they are often marked with an index  $L$  to indicate a size  $2^L$ .

$\mathbf{O}_L(x)$  is the operator which adds a unit on site  $i = 1$  and relaxes the entire lattice of size  $L$ . It is a matrix of size  $2^L \times 2^L$  and defined as

$$\mathbf{O}_L(x) = \epsilon \mathbf{1}^{\otimes L} + \delta (S \otimes \mathbf{1}^{\otimes(L-1)} + xT \otimes \mathbf{O}_{L-1}(x) + x^2U \otimes \mathbf{O}_{L-1}^2(x)) \quad (10)$$

again with a dissipation rate  $\epsilon$ , leaving the state unchanged. The bracket multiplied by  $\delta$  consists of three terms: the first term charges the site without toppling and leaves the rest of the system unchanged by operating with the identity  $\mathbf{1}^{\otimes(L-1)}$ . The second term corresponds to a single toppling, which charges the remaining system of size  $L - 1$  once. This term is derived using the identity

$$(T \otimes \mathbf{1}^{\otimes(L-1)})(\mathbf{1} \otimes \mathbf{O}_{L-1}(x)) = T \otimes \mathbf{O}_{L-1}(x). \quad (11)$$

The third term is a double toppling of the site, giving rise to a double charge of the remaining system.

The Abelian property mentioned above (section 2.1) can be expressed as the commutator for two charges on a system of size  $L$ , one at site  $i = 1$ , the other one at site  $1 + L - L'$  with  $L'$  being the size of the subsystem starting from the site receiving the second charge,

$$\mathbf{O}_L(x)(\mathbf{1}^{\otimes(L-L')} \otimes \mathbf{O}_{L'}(x)) = (\mathbf{1}^{\otimes(L-L')} \otimes \mathbf{O}_{L'}(x))\mathbf{O}_L(x) \tag{12}$$

where of course  $L \geq L'$ . The tensor multiplication used on  $\mathbf{O}_{L'}$  and also in (11) ensures that both matrices have the same rank; they are ‘filled with identity’ where they do not act. Equation (12) simply states that it does not matter for the statistics whether the leftmost site of a right subsystem of size  $L'$  in a system of size  $L$  is charged first, followed by the leftmost site of the entire system, or vice versa. Due to the asymmetry in the dynamics, it is clear that a system of size  $L$ , initially charged at site  $i$ , has the same statistics as a system of size  $L - i + 1$ , charged at its leftmost site. It might be interesting, however, to formally prove equation (12), which should easily be feasible using established methods [18, 15].

The distribution of states at time  $t$  is the vector  $|P_t\rangle_L$ , which has rank  $2^L$ , each row corresponding to the probability for the system to be in the state encoded by that row. The encoding follows from the row ordering convention introduced above and the use of the tensor product in (10).

For  $x = 1$  the operator  $\mathbf{O}_L(x)$  is simply the Markov matrix acting on  $|P_t\rangle_L$ , producing the distribution of states at time  $t + 1$  [17]

$$|P_{t+1}\rangle_L = \mathbf{O}_L(1)|P_t\rangle_L. \tag{13}$$

There exists at least one eigenvector with eigenvalue 1, which is therefore a stationary distribution. If the eigenvectors represent a complete basis and the modulus of all other eigenvalues is less than unity, this stationary distribution is unique and reached by *any* initial distribution. The stationary distribution, denoted by  $|0\rangle_L$ , is the focus of the following calculations. It is shown below that it is unique.

One very important bra-eigenvector with eigenvalue 1 of  $\mathbf{O}_L(x = 1)$  is

$$\langle 0|_L \equiv \underbrace{(1, 1, \dots, 1)}_{2^L \text{ times}} \tag{14}$$

by normalization. As has been indicated above, for general  $x$ , the operator  $\mathbf{O}_L(x)$  becomes a moment generating function of the avalanche size, if sandwiched between  $\langle 0|_L$  and the stationary distribution:

$$Q_{L,n}(x; \epsilon) \equiv \langle 0|_L \mathbf{O}_L^n(x) |0\rangle_L. \tag{15}$$

This can be seen from (10) containing an  $x$  for every toppling. When the operator acts on a distribution, for each transition from one state to another a power of  $x$  corresponding to the number of topplings enters and is multiplied by the probability to be in the initial state (given by the initial distribution) and the transition probability given by the transition matrix. The function  $Q_{L,n}(x; \epsilon)$  for general  $n$  is then the generating function for avalanches caused by  $n = 1, 2, \dots$  initial kicks. In particular

$$\langle s^m \rangle_L = \left( x \frac{d}{dx} \right) \Big|_{x=1}^m Q_{L,1}(x; \epsilon). \tag{16}$$

The aim of the following calculations is to find the generating function  $Q_{L,1}(x; \epsilon)$  or at least the moments generated by it.

### 3.1. General eigenvectors and eigenvalues of $\mathbf{O}_L(x)$

It would be very helpful if  $\mathbf{O}_L(x)$  could be written in the form

$$\mathbf{O}_L(x) = \sum_{i=0}^{2^L-1} |i(x)\rangle_L \lambda_{L,i} \langle i(x)|_L \quad (17)$$

where  $\langle i(x)|_L$  denote the left hand and  $|i(x)\rangle_L$  the right hand eigenvectors of  $\mathbf{O}_L(x)$  with eigenvalues  $\lambda_{L,i}(x)$  and  $i = 0, \dots, 2^L - 1$ . *A priori* it is not clear whether these vectors actually exist. In the following they are constructed and it is shown that setting  $\epsilon = 0$  leads to fundamental problems.

Assuming that  $|i\rangle_{L-1}$  is an eigenvector with eigenvalue  $\lambda_{L-1,i}$  of  $\mathbf{O}_{L-1}(x)$ , the definition of  $\mathbf{O}_L(x)$ , equation (10), gives for an arbitrary vector  $|e\rangle_1$

$$\mathbf{O}_L(x) (|e\rangle_1 \otimes |i\rangle_{L-1}) = [\{\epsilon \mathbf{1} + \delta (S + xT\lambda_{L-1,i} + x^2U\lambda_{L-1,i}^2)\} |e\rangle_1] \otimes |i\rangle_{L-1} \quad (18)$$

where  $|e\rangle_1$  contains two elements such that  $|e\rangle_1 \otimes |i\rangle_{L-1}$  is a vector of  $2^L$  elements. The matrix in the curly brackets is simply  $\mathbf{O}_1(x\lambda_{L-1,i})$ . So, if  $|e\rangle_1$  is either  $|e_\lambda(x\lambda_{L-1,i})\rangle$  or  $|e_\mu(x\lambda_{L-1,i})\rangle$  from (8), then  $|e\rangle_1 \otimes |i\rangle_{L-1}$  is an eigenvector of  $\mathbf{O}_L(x)$  with eigenvalues  $\epsilon + \delta(x\lambda_{L-1,i})$  or  $\epsilon$ . Thus, based on (8), one can write the eigenvectors of  $\mathbf{O}_L(x)$  as

$$\begin{aligned} |i\rangle_L &= |e_\lambda(x\lambda_{L-1,i})\rangle \otimes |i\rangle_{L-1} \\ \langle i|_L &= \langle e_\lambda(x\lambda_{L-1,i})| \otimes \langle i|_{L-1} \\ |i + 2^{L-1}\rangle_L &= |e_\mu(x\lambda_{L-1,i})\rangle \otimes |i\rangle_{L-1} \\ \langle i + 2^{L-1}|_L &= \langle e_\mu(x\lambda_{L-1,i})| \otimes \langle i|_{L-1} \end{aligned} \quad (19)$$

and the eigenvalues as

$$\lambda_{L,i} = \epsilon + x\delta\lambda_{L-1,i} \quad \lambda_{L,i+2^{L-1}} = \epsilon \quad (20)$$

both with  $i = 0, 1, \dots, 2^{L-1} - 1$ . To start the hierarchy, one defines

$$|0\rangle_1 = |e_\lambda(x)\rangle \quad \langle 0|_1 = \langle e_\lambda(x)| \quad |1\rangle_1 = |e_\mu(x)\rangle \quad \langle 1|_1 = \langle e_\mu(x)| \quad (21)$$

and the eigenvalues as

$$\lambda_{1,0} = \epsilon + x\delta \quad \lambda_{1,1} = \epsilon. \quad (22)$$

Now it is clear why the quantity  $\epsilon$  was necessary: for  $\epsilon = 0$  all but one eigenvalues vanish, which can be seen from the hierarchy of eigenvalues obtained by iterating (20). Therefore, if the vanishing eigenvalue of  $L - 1$  is plugged into  $\langle e_\lambda|$  or  $\langle e_\mu|$  according to (19), the result is undefined, as can be seen from (8), so that the bra-eigenvectors cease to exist.

The fact that all but one eigenvalues vanish for  $\epsilon = 0$  is very deceptive. Assuming that any initial condition  $|P\rangle$  can be written in terms of the eigenvectors of  $\mathbf{O}_L(1)$ , say  $\sum a_i |i\rangle$ , this suggests  $\mathbf{O}_L(1)|P\rangle = |0\rangle$ . This, however, is wrong, because for vanishing  $\epsilon$  the operator  $\mathbf{O}_L(x)$  cannot be written in the form (17) for  $L > 1$ . And it must be wrong, because, for example, kicking an empty system once cannot make it produce the stationary distribution.

If the eigenvectors of  $\mathbf{O}_{L-1}$  are linearly independent, then one can show, using the construction (19), that the eigenvectors of  $\mathbf{O}_L$  are linearly independent as well, provided that  $|e_\lambda(x\lambda_{L-1,i})\rangle$  and  $|e_\mu(x\lambda_{L-1,i})\rangle$  are linearly independent. This is not the case for  $\epsilon = 0$  (see the ket vectors in (8) with  $x = 0$ ) and this is the basic reason why  $\epsilon \neq 0$  is needed for the time being. However, for any  $\epsilon \neq 0$  one can apparently construct a diagonalizing matrix for  $\mathbf{O}_L$ . Thus, it can be written in the form (17). Especially, the eigenvectors have the property (by induction)

$$\langle i(x)|j(x)\rangle_L = \delta_{i,j} \quad (23)$$

and as all  $2^L$  eigenvectors are linearly independent, they must span the whole space so that

$$\sum_{i=0}^{2^L-1} |i\rangle_L \langle i|_L = \mathbf{1}. \tag{24}$$

In the form (17) the operator can now be applied to a stationary distribution to give

$$Q_{L,n}(x; \epsilon) = \sum_{i=0}^{2^L-1} \langle 0|i(x)\rangle_L \lambda_{L,i}^n \langle i(x)|0\rangle_L. \tag{25}$$

*3.1.1. The stationary distribution.* From (19) the stationary distribution can be derived immediately. It is the eigenvector with eigenvalue 1 of  $\mathbf{O}_L(1)$ . Setting  $x = 1$  in (20) it is clear that  $\lambda_{L,i} = 1$  requires a  $\lambda_{L-1,j} = 1$ , which, together with (22), gives the unique  $\lambda_{L,0} = 1$  provided that  $\epsilon < 1$ . If  $\epsilon = 1$ , then *all* eigenvalues are 1, but still all eigenvectors are linearly independent and therefore span the entire space, so that *all* initial distributions are stationary. This is not surprising because  $\epsilon = 1$  simply means that any added particle immediately dissipates from the system, so that adding a particle is in fact just the identity operation.

For  $0 \leq \epsilon < 1$  the stationary distribution is unique and all other eigenvalues have modulus less than 1. The eigenvector corresponding to eigenvalue  $\lambda_{L,0} = 1$  is, according to (19),

$$\langle 0|_L = \langle e_\lambda(1)|^{\otimes L} = (1, 1)^{\otimes L} \tag{26a}$$

$$|0\rangle_L = |e_\lambda(1)\rangle^{\otimes L} = \begin{pmatrix} p \\ q \end{pmatrix}^{\otimes L} \tag{26b}$$

which is consistent with the notation for the stationary distribution and the normalization eigenvector introduced in (15) and (14). The last line, equation (26b), indicates that the stationary state is a product measure, i.e. a state at one site does not depend on the state on any other site. In fact the spatial correlation function of sites  $\{i_1, i_2, \dots\}$  can easily be calculated by ‘dressing’ the states of the sites by appropriate powers of a variable  $x_i$ , in order to obtain the generating function of the correlators. The function

$$C(x_1, x_2, \dots, x_L) = \langle 0|_L \begin{pmatrix} px_1 \\ qx_1^{-1} \end{pmatrix} \begin{pmatrix} px_2 \\ qx_2^{-1} \end{pmatrix} \cdots \begin{pmatrix} px_L \\ qx_L^{-1} \end{pmatrix} = \prod_i^L (px_i + qx_i^{-1}) \tag{27}$$

is the generating function of the state correlators, where state 1 stands for  $z = 1$  and state  $-1$  for  $z = 2$ . The states have the useful property that the joint contribution of two sites is 1 if both sites are in the same state and  $-1$  otherwise. The average state is obtained from

$$x_i \frac{d}{dx_i} \Big|_{x_1, \dots, x_L=1} \ln(C(x_1, x_2, \dots, x_L)) = \frac{d}{dx_i} \Big|_{x_1, \dots, x_L=1} C(x_1, x_2, \dots, x_L) = p - q. \tag{28}$$

Correspondingly, the connected two-point correlation function of sites  $i$  and  $j$  is given by

$$x_j \frac{d}{dx_j} x_i \frac{d}{dx_i} \Big|_{x_1, \dots, x_L=1} \ln(C(x_1, x_2, \dots, x_L)) = \begin{cases} 4pq & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{29a}$$

$$\tag{29b}$$

This confirms the absence of correlations and is fully consistent with the expected variance of the state.



### 3.2. The hierarchy of generating functions

Using (25), one can now calculate the generating function  $Q_{L,n}(x; \epsilon)$ , by plugging the hierarchy of eigenvectors (19) and eigenvalues (20) into (25) and using the properties of  $\otimes$ , see equation (4). For  $n = 1$  it is

$$Q_{L,1}(x; \epsilon) = \sum_{i=0}^{2^{L-1}-1} x \delta \lambda_{L-1,i} \langle 0|i \rangle_{L-1} \langle 0|e_\lambda(x\lambda_{L-1,i})\rangle_1 \langle i|0 \rangle_{L-1} \langle e_\lambda(x\lambda_{L-1,i})|0 \rangle_1 \\ + \sum_{i=0}^{2^L-1} \epsilon \langle 0|i(x) \rangle_L \langle i(x)|0 \rangle_L$$

where the term proportional to  $\epsilon$  comes from the  $\epsilon$  in every  $\lambda_{L,i}$  (see equation (20)). From (24) it is clear that the last sum gives 1. The two projections give

$$\langle 0|e_\lambda(x\lambda_{L-1,i})\rangle_1 = x\lambda_{L-1,i}p + q \quad \text{and} \quad \langle e_\lambda(x\lambda_{L-1,i})|0 \rangle_1 = \frac{p}{x\lambda_{L-1,i}} + q$$

so that

$$Q_{L,1}(x; \epsilon) = \epsilon + \sum_{i=0}^{2^{L-1}-1} x \delta \lambda_{L-1,i} \langle 0|i \rangle_{L-1} \langle i|0 \rangle_{L-1} \left( p^2 + q^2 + pq \left( x\lambda_{L-1,i} + \frac{1}{x\lambda_{L-1,i}} \right) \right) \quad (30a)$$

$$= \epsilon + \delta((p^2 + q^2)x Q_{L-1,1}(x; \epsilon) + pq(x^2 Q_{L-1,2}(x; \epsilon) + Q_{L-1,0}(x; \epsilon))) \quad (30b)$$

where equation (25) has been used in the last line. Of course, the generating function  $Q_{L,n}(x; \epsilon)$  is defined, (15), for all  $\epsilon$  and therefore one can take the limit  $\epsilon \rightarrow 0$ . This limit should not cause any problems, as  $\epsilon$  has only been used to construct the eigenvectors. In fact, the limit must be identical to setting  $\epsilon = 0$  in (30), as can be shown from (30) by induction in  $L$ . This finally gives

$$Q_{L,1}(x; 0) = (p^2 + q^2)x Q_{L-1,1}(x; 0) + pq(x^2 Q_{L-1,2}(x; 0) + Q_{L-1,0}(x; 0)) \quad (31)$$

where  $Q_{L-1,0}(x; \epsilon) = 1$  by equation (15), consistent with equation (25). In fact, the calculation above can be generalized:

$$Q_{L,n}(x; \epsilon) = \sum_{i=0}^{2^{L-1}-1} \sum_{j=1}^n \binom{n}{j} \epsilon^{n-j} (x\delta\lambda_{L-1,i})^j \\ \times \langle 0|i_{L-1} \rangle \langle 0|e_\lambda(x\lambda_{L-1,i})\rangle_1 \langle i|0 \rangle_{L-1} \langle e_\lambda(x\lambda_{L-1,i})|0 \rangle_1 \\ + \sum_{i=0}^{2^L-1} \epsilon^n \langle 0|i(x) \rangle_L \langle i(x)|0 \rangle_L. \quad (32)$$

Again, all sums can be written in terms of  $Q_{L-1,n}(x; \epsilon)$  plus  $\epsilon^n$ :

$$Q_{L,n}(x; \epsilon) = \epsilon^n + \sum_{j=1}^n \binom{n}{j} \delta^j \epsilon^{n-j} ((p^2 + q^2)x^j Q_{L-1,j}(x; \epsilon) \\ + pqx^{j+1} Q_{L-1,j+1}(x; \epsilon) + pqx^{j-1} Q_{L-1,j-1}(x; \epsilon)). \quad (33)$$

For vanishing dissipation this simplifies to the central result

$$Q_{L+1,n}(x; 0) = x^n (\tilde{D} Q_{L,n}(x; 0) + D(x Q_{L,n+1}(x; 0) + x^{-1} Q_{L,n-1}(x; 0))) \quad (34)$$

with  $D = pq$  and  $\tilde{D} = p^2 + q^2 = 1 - 2D$ . Equation (34) is closely related to a diffusion equation. The boundary conditions are  $Q_{L,0}(x; \epsilon) \equiv 1$  for  $L \geq 1$  as mentioned above and  $Q_{L=0,n}(x; \epsilon) \equiv 1$ . The latter comes from a direct evaluation of (25) for  $L = 1$ , which is identical to (33) for  $Q_{L=0,n}(x; \epsilon) \equiv 1$ .

3.3. Solving  $Q_{L,n}$

There is no general solution for (34) known to the author. However, one can solve it order by order in derivatives by  $x$  at  $x = 1$ , i.e. calculate every individual moment, see (16). In the following, the notation

$$Q_{L+1,n} = Q_{L+1,n}(1; 0) \tag{35a}$$

$$Q'_{L+1,n} = \left. \frac{d}{dx} \right|_{x=1} Q_{L+1,n}(x; 0) \tag{35b}$$

etc is used. One finds for  $n \geq 1$

$$Q_{L+1,n} = \tilde{D}Q_{L,n} + D(Q_{L,n+1} + Q_{L,n-1}) \tag{36}$$

which is solved with the boundary conditions introduced above by  $Q_{L,n} = 1$ . Of course, this is just normalization. Using this result the next derivative is

$$Q'_{L+1,n} = n + \tilde{D}Q'_{L,n} + D(Q'_{L,n+1} + Q'_{L,n-1}) \tag{37}$$

with the boundary conditions  $Q'_{L+1,n=0} = 0$  and  $Q'_{L=0,n} = 0$ . The solution of (37) can easily be guessed as

$$Q'_{L,n} = nL. \tag{38}$$

This is not surprising, because it says that the average number of topplings occurring in the system per  $n$  kicks is  $nL$ . That is obviously true, because every unit added must leave the system by travelling through the entire lattice.

The next order is the first non-trivial one. The difference equation then reads

$$Q''_{L+1,n} = (n^2 + 2D)(2L + 1) - n + \tilde{D}Q''_{L,n} + D(Q''_{L,n+1} + Q''_{L,n-1}). \tag{39}$$

Introducing

$$Q''_{L+1,n} = S_{L+1,n} + \tilde{Q}''_{L+1,n} \tag{40}$$

with

$$S_{L,n} = \sum_{i=0}^{L-1} (n^2 - n + 2in^2) = -nL + n^2L^2 \tag{41}$$

which has the useful property  $S_{L+1,n} - S_{L,n} = -n + 2n^2L + n^2$  and  $S_{L,n+1} + S_{L,n-1} = 2S_{L,n} + 2L^2$ , removes the  $n$  dependence from the source term,

$$\tilde{Q}''_{L+1,n} = 2D(L + 1)^2 + \tilde{D}\tilde{Q}''_{L,n} + D(\tilde{Q}''_{L,n+1} + \tilde{Q}''_{L,n-1}). \tag{42}$$

The solution is therefore simply

$$\tilde{Q}''_{L,n} = \sum_{l=1}^L 2Dl^2 \sum_{m=1}^{\infty} \psi_{L-l,n,m} \tag{43}$$

with the propagator  $\psi_{L,n,m}$ , obeying

$$\psi_{L+1,n,m} = \tilde{D}\psi_{L,n,m} + D(\psi_{L,n+1,m} + \psi_{L,n-1,m}) \tag{44}$$

with boundary condition  $\psi_{L,n=0,m} = 0$  (from  $\tilde{Q}''_{L,n=0} = 0$ ) and initial condition  $\psi_{L=0,n,m} = \delta_{n,m}$ . This propagator describes the density of discrete random walkers at  $n > 0$  starting at  $m$ . The boundary condition can be satisfied by a mirror charge trick applied to  $\psi_{L,n}^0$ , which obeys (44) in the whole plane  $n$ , where  $\psi_{L,n}^0 = \psi_{L,-n}^0$ , so that (43) becomes

$$\tilde{Q}''_{L,n} = \sum_{l=1}^L 2Dl^2 \sum_{m=1}^{2n} \psi_{L-l,n-m}^0. \tag{45}$$

The propagator is constructed by Fourier transforming (44), which gives

$$\tilde{\psi}_{L,k}^0 = [(p e^{ik/2} + q e^{-ik/2})(p e^{-ik/2} + q e^{ik/2})]^L \quad (46)$$

and therefore

$$\psi_{L,n}^0 = \sum_{m=0}^L \binom{L}{m} \binom{L}{m-n} p^{2L-2m+n} q^{2m-n}. \quad (47)$$

The well-known solution of the diffusion equation on the lattice

$$\binom{2L}{L+n} 2^{-2L} \quad (48)$$

is obtained for  $p = q = 1/2$  from (47) using ‘Vandermode’s convolution’ [19]

$$\sum_{m=0}^L \binom{L}{m} \binom{L}{m-n} = \binom{2L}{L+n}. \quad (49)$$

For  $n = 1$  (45) reads

$$\tilde{Q}_{L,1}'' = \sum_{l=0}^{L-1} 2D(L-l)^2 \sum_{m=0}^l \binom{l}{m} p^{2l-2m} q^{2m} \left( \binom{l}{m} + \binom{l}{m+1} \frac{q}{p} \right). \quad (50)$$

In order to analyse the asymptotic behaviour for  $L \rightarrow \infty$ , one writes

$$\tilde{Q}_{L,1}'' = \sum_{l=0}^{L-1} 2D(L-l)^2 \phi^*(l) \quad (51)$$

with

$$\phi^*(l) = \sum_{m=0}^l \binom{l}{m} p^{2(l-m)} q^{2m} \left( \binom{l}{m} + \binom{l}{m+1} \frac{q}{p} \right). \quad (52)$$

For large  $l$  the binomials can be approximated by a Gaussian

$$\binom{l}{m} p^{l-m} q^m \rightarrow \frac{1}{\sqrt{2\pi l p q}} \exp\left(-\frac{(m-lq)^2}{2l p q}\right) \quad (53)$$

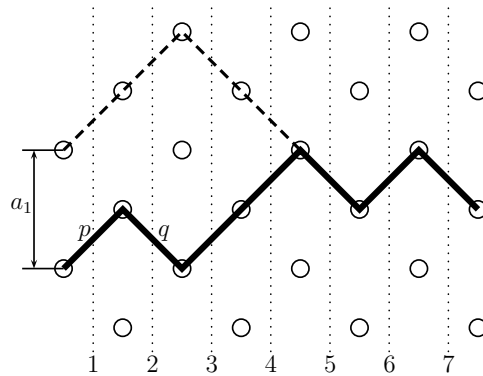
and writing (52) as an integral over  $m$ , one arrives at

$$\tilde{Q}_{L,1}'' \rightarrow \sum_{l=0}^{L-1} 2D(L-l)^2 \frac{1}{\sqrt{4\pi l p q}} (\mathcal{E}(\sqrt{l p/q}) + \mathcal{E}(\sqrt{l q/p})) \quad (54)$$

where  $\mathcal{E}(x) \equiv 2 \int_0^x dz \exp(-z^2)/\sqrt{\pi}$ . Evaluating the sum as an integral, in leading order this turns out to be

$$\tilde{Q}_{L,1}'' \rightarrow \frac{32}{15\sqrt{\pi}} \sqrt{p q} L^{5/2} \quad (55)$$

which is according to (40) also the leading order of  $Q_{L+1,n}''$  and therefore the leading order of  $\langle s^2 \rangle$ , see (16) with (38). This is perfectly confirmed by numerical simulations of the model. The two exponents  $\gamma_1 = 1$  (see equation (38)) and  $\gamma_2 = 5/2$  (55) lead together with (3) to  $\tau = 4/3$  and  $D = 3/2$ .



**Figure 1.** The thick, full line shows the configuration of the lattice after an avalanche has passed through. Each up- or down-pointing segment corresponds to a single site, the position label of which is shown under the dotted line. A segment pointing upwards corresponds to a site being in state  $z = 1$  (with probability  $p$ , see equation (26b)), a segment pointing downwards corresponds to state  $z = 2$  (probability  $q$ ), as indicated. The dashed line corresponds to a 'toppling trajectory' as explained in the text.

#### 4. Reaction–diffusion mapping

It is possible to map the model onto a very simple reaction–diffusion process of the form  $A + A \rightarrow A$  [20]. To this end, the configuration of the lattice is described by the thick line shown in figure 1. The line consists of segments, which can either point up or down by an angle of  $45^\circ$ . If the line corresponding to the  $i$ th site goes up, it indicates that the  $i$ th site is in state  $z = 1$ , otherwise the line goes down indicating the state of the site to be  $z = 2$ . According to (26b) the configuration of the lattice (in the stationary state) after an avalanche is a product state, where a site is in state  $z = 1$  with probability  $p$  and in state  $z = 2$  with probability  $q$ . Thus, the thick line is in fact the trajectory of a random walker with drift corresponding to the difference  $p - q$ .

The avalanche itself, on the other hand, is a random walk with the same probabilities. One can see that by considering the activity  $a_i$ , which is the number of charges received at site  $i$  during an update sweep as described in section 2.1. From site to site, the activity can either remain constant or change by 1 up or down. Apparently,  $a_1 = 1$  is the driving. If a site receives  $a_i$  charges and changes state by  $\Delta z_i = z_i(t) - z_i(t + 1)$ , then its right neighbour receives  $a_{i+1} = a_i + \Delta z_i$  charges. If  $\Delta z_i = 0$ , then the vertical distance between two consecutive configuration trajectories (as shown by the thick and dashed lines in figure 1) does not change. If, however, the new configuration of site  $i$  has an increased value  $z_i(t + 1) > z_i(t)$ , the activity goes down, because  $\Delta z_i < 0$ . The only way to increase  $z_i$  is to go from state 1 to state 2, i.e. the line segment of the former configuration points up (probability  $p$ ), while the line segment of the new configuration points down (probability  $q$ ), so that the gap between the two trajectory decreases, see figure 1 at the dotted line 3. Similarly, if the activity goes up, then  $\Delta z_i > 0$  and the gap increases.

After the activity vanishes, the profile of the new configuration remains unchanged compared to the former, i.e. the gap between the two configurations is a constant. In fact, if the gap was initially 1 and goes up and down by 1 as described above, then the gap will be 0 as soon as the activity vanishes. This is exactly what is shown in figure 1: *the thick dashed line shows the new configuration and its distance to the old configuration is the activity during the avalanche.* This avalanche occurs within the configuration shown as a thick line, initiated by a

single kick. Initially the gap is  $a_1 = 1$ . If the dashed line went down immediately on site 1, the site would have ‘absorbed’ the initial unit and would be in state  $z = 2$  (i.e. a segment pointing down). Instead, in the example, it goes up twice; first just like in the old configuration so that the activity does not increase, and then in the opposite direction to the old configuration so that the activity increases by 1. On sites 3 and 4 it goes down twice; the toppling on site 4 is particularly interesting. Here, initially the activity is 1, i.e. the site has received 1 unit. But the site is in state  $z = 1$ , so it absorbs the unit with probability  $q$ , corresponding to the probability of the dashed line segment to point downwards.

The activity is measured half a unit left of each site as the distance between the old and new trajectories, which, in turn, is measured in such units that the vertical distance between two circles (in figure 1) is 1. The reason for the shift is that one wants to measure how many charges have *arrived* at a site, not affected by the value of the resulting activity.

To repeat this important point, the trajectory of an avalanche becomes the configuration for the next avalanche, i.e. the thick dashed line in figure 1 becomes the thick solid line for the next avalanche.

One can calculate the probability of the changes of activity explicitly: the new segment goes up with probability  $p$  and down with probability  $q$ , the same applies to the old segment. Thus, they point in the same direction (no change of activity) with probability  $p^2 + q^2$ , the gap widens with probability  $pq$  and shrinks with  $qp$ . Hence, the gap between the two trajectories is in fact a symmetric random walk, even though the individual trajectories might have a bias, according to  $p - q$ .

As described above (see section 2), the avalanche size is measured as the number of topplings. For convenience, one can define it as the number of charges, which makes hardly any difference, because the number of topplings of site  $i$  is identical to the number of charges on site  $i + 1$ , unless  $i = L$ , simply because there is no site  $L + 1$ ; similarly for  $L = 1$ .

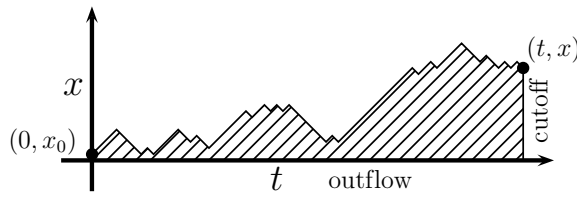
In the following, we will consider the number of charges as the avalanche size, because the total number of charges is simply the area between two of those trajectories described above, namely the sum over all activities. From this it is also clear that the avalanche size *is* actually uniquely determined by the initial and the final configurations, with initial activity  $a_1 = 1$ .

#### 4.1. Relation to other models

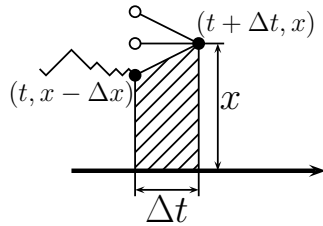
Before the above identification of the process as a random walker is cast into an continuum problem and subsequently solved, it is worth pointing out other models which are closely linked to the present one.

*4.1.1. Anisotropic BTW model.* Dhar and Ramaswamy [21] developed an anisotropic variant of the well-known BTW sandpile model [1], which is now known as the directed sandpile model. This model, however, is situated on a  $(1 + 1)$ -dimensional lattice and the annihilating random walkers represent the contours of the compact area covered by an avalanche. The randomness here comes solely from the randomness of whether a site charged by particles from toppling sites topples in turn. An equivalence to a variant of directed percolation [22] has already been pointed out in [21], see also [23].

Kloster, Maslov and Tang [24] have studied a stochastic directed sandpile model, which was originally proposed by Pastor-Satorras and Vespignani [25]. This model is closely related to the one presented in this paper, even though it is also situated on a  $(1 + 1)$ -dimensional lattice. For its one-dimensional variant [26], the authors find the same exponents by scaling arguments. The mapping to the two-dimensional reaction–diffusion process presented above,



**Figure 2.** The area under the trajectory (hatched) is the avalanche size. The two filled circles mark the starting point  $(0, x_0)$  and the end point  $(t, x)$ .



**Figure 3.** A new segment (hatched area) is added to the currently considered path, increasing all areas in the ensemble  $\{s_i(t, x - \Delta x; x_0)\}$  by  $x\Delta t + \mathcal{O}(\Delta t \Delta x)$  and producing a new ensemble  $\{s_i(t + \Delta t, x; x_0)\}$ . The example shown corresponds to (56c), which starts at  $(t, x - \Delta x)$ . The starting points of other contributions are shown as empty circles. The coordinates of the two black points are given in the form  $(t, x)$ .

questions their assertion that their model is in a different universality class than the model by Dhar and Ramaswamy.

For these models it is fairly obvious how to extend them systematically to higher dimensions. Using scaling arguments in conjunction with some simplifying assumptions, Paczuski and Bassler [27] arrive at a general expression for the value of the exponents of this model in higher dimensions. Unfortunately, it is not so clear how to generalize the model studied in this paper to higher dimensions, because it is unclear how to generalize the driving and what boundary conditions to apply.

### 5. Continuum solution

Having mentioned already the mapping to an annihilating random walk, the continuum description is straight-forward. To this end, the quantity  $\psi_n(t, x; x_0)$  is introduced. It quantifies the properties of a random walker along an absorbing wall. For  $n = 0$  it is the probability density of random walkers at time  $t$  and height  $x$  over the absorbing wall, starting at height  $x_0$ , which is  $x_0 = 1$  for a single initial kick. Here,  $t$  takes on the rôle of the horizontal (continuous) position between  $t = 0$  and  $t = L$  in a picture like figure 1. To motivate the following calculation, one imagines a large set of trajectories of random walkers along the absorbing wall from  $t = 0, x = x_0$  to  $t$  and  $x$ . The set of areas under the trajectories, as exemplified in figure 2, is then  $\{s_i(t, x; x_0)\}$ , where  $i$  is indexing the elements in the set.  $\langle \{s^n(t, x; x_0)\} \rangle$  is the average of the  $n$ th moment over this set. Now one can express the time evolution of this average as the sum of three contributions of the three processes of up, down or straight movement of the random walker. Thus, up to terms of order  $\Delta t \Delta x$  (see caption of figure 3)

$$\begin{aligned} &\psi_0(t + \Delta t, x; x_0) \langle \{s^n(t + \Delta t, x; x_0)\} \rangle \\ &= pq \psi_0(t, x + \Delta x; x_0) \langle \{(s(t, x + \Delta x; x_0) + x \Delta t)^n\} \rangle \end{aligned} \tag{56a}$$

$$+ (p^2 + q^2)\psi_0(t, x; x_0)\{(s(t, x; x_0) + x\Delta t)^n\} \quad (56b)$$

$$+ pq\psi_0(t, x - \Delta x; x_0)\{(s(t, x - \Delta x; x_0) + x\Delta t)^n\} \quad (56c)$$

where each term corresponds to a process like the one shown in figure 3. The multiplication by  $\psi_0(t, x; x_0)$  is necessary in order to weight each of the ensembles for each contribution properly. For example, a much larger contribution might come from below, even though on average the moment at this position is smaller than at the other positions.

Defining

$$\psi_n(t, x; x_0) \equiv \psi_0(t, x; x_0)\{s(t, x; x_0)^n\} \quad (57)$$

one finds in the continuum limit of (56) (keeping  $D\Delta t/\Delta x^2$  constant)

$$\partial_t \psi_n(t, x; x_0) = D\partial_x^2 \psi_n(t, x; x_0) + xn\psi_{n-1}(t, x; x_0) \quad (58)$$

where  $D = pq$  again<sup>1</sup>. The boundary conditions for  $n = 0$  are observed immediately and transferred to  $\psi_n$  using (57) by noting that  $\{s(t, x; x_0)^n\}$  is non-divergent, so

$$\lim_{t \rightarrow 0} \psi_n(t, x; x_0) = \delta_{n,0}\delta(x - x_0) \quad (59a)$$

$$\psi_n(t, 0; x_0) = 0 \quad (59b)$$

and the PDE (58) is to be solved for  $x \in [0, \infty[$ .

The avalanche sizes are measured from avalanche trajectories which have died out or reached the end of the system. Thus, the averages measured in the model are taken from the random walkers which have reached the absorbing wall or did not do so until a cutoff time  $t$ . Therefore the  $n$ th moment observed is

$$\langle s^n \rangle(t; x_0) = \int_0^t dt' j_n(t'; x_0) + \int_0^\infty dx' \psi_n(t, x'; x_0) \quad (60)$$

where the first integral runs over the ‘outflow’,  $j_n(t, x = 0; x_0) \equiv D\partial_x|_{x=0}\psi_n(t, x; x_0)$  and the second over the contributions at cutoff time (see marks in figure 2).  $\langle s^n \rangle(t; x_0)$  denotes the  $n$ th moment of the avalanche size (measured as the number of charges) for a system of size  $t$  starting with  $x_0$  initial charges. Using (58) one has

$$\langle s^n \rangle(t; x_0) = \int_0^t dt' \int_0^\infty dx' x'^n \psi_{n-1}(t', x'; x_0). \quad (61)$$

The dimensionless form of  $\psi$  is given by

$$\psi_n(x, t; x_0) = \frac{1}{x_0} \left( \frac{x_0^3}{D} \right)^n \tilde{\psi}_n(y, \tau) \quad (62)$$

with  $y = x/x_0$  and  $\tau = t/(x_0^2/D)$ . The propagator  $G(y, \tau; y_0)$  is easily obtained from a mirror-charge trick,

$$G(y, \tau; y_0) \equiv \frac{1}{\sqrt{4\tau\pi}} \left( \exp\left(-\frac{(y-y_0)^2}{4\tau}\right) - \exp\left(-\frac{(y+y_0)^2}{4\tau}\right) \right) \quad (63)$$

and  $\tilde{\psi}_0(y, \tau) = G(y, \tau; 1)$ , i.e.

$$\tilde{\psi}_0(y, \tau) = \frac{1}{\sqrt{\tau\pi}} \exp\left(-\frac{y^2+1}{4\tau}\right) \sinh\left(\frac{y}{2\tau}\right). \quad (64)$$

<sup>1</sup> It is interesting to note that this can be written using a generating function  $\Psi(t, x; x_0, \xi)$  with  $\partial_t \Psi = x\xi\Psi + D\partial_x^2 \Psi$ , so that indeed  $\left. \frac{d^n}{d\xi^n} \right|_{\xi=0} \Psi(t, x; x_0, \xi) = \psi_n(t, x; x_0)$ .

One might be inclined to transfer the problem into  $k$ -space, which, however, does not simplify the problem because of the boundary condition (59b). The expression

$$\tilde{\psi}_n(y, \tau) = \int_0^\tau d\tau' \int_0^\infty dy' ny' \tilde{\psi}_{n-1}(y', \tau') G(y, \tau - \tau'; y') \tag{65}$$

is the formal solution. Rescaling the arguments of  $\tilde{\psi}_n$  by powers of  $\mu$  one finds

$$\tilde{\psi}_n(\sqrt{\mu}y, \mu\tau) = \mu^{3/2} \int_0^\tau d\tau' \int_0^\infty dy' ny' \tilde{\psi}_{n-1}(\sqrt{\mu}y', \mu\tau') G(y, \tau - \tau'; y'). \tag{66}$$

From  $\tilde{\psi}_{n-1}(\sqrt{\mu}y, \mu\tau) = \mu^{\alpha_{n-1}} \tilde{\psi}_{n-1}(y, \tau)$ , then follows  $\tilde{\psi}_n(\sqrt{\mu}y, \mu\tau) = \mu^{\alpha_{n-1}+3/2} \tilde{\psi}_n(y, \tau)$ . Thus, starting with  $\tilde{\psi}_0(\sqrt{\mu}y, \mu\tau) = \mu^{\alpha_0} \tilde{\psi}_0(y, \tau)$  one has apparently

$$\tilde{\psi}_n(\sqrt{\mu}y, \mu\tau) = \mu^{\frac{3}{2}n+\alpha_0} \tilde{\psi}_n(y, \tau). \tag{67}$$

Unfortunately the scaling behaviour of  $\tilde{\psi}_0$  is a bit more complicated. Nevertheless, it can be expanded for large  $\mu$ , or actually large  $\mu\tau$ , as

$$\tilde{\psi}_0(\sqrt{\mu}y, \mu\tau) = \frac{1}{\mu} \frac{1}{\sqrt{\tau\pi}} \exp\left(-\frac{y^2}{4\tau}\right) \left(\frac{y}{2\tau} + \frac{1}{\mu} \left(\frac{y^3}{48\tau^3} - \frac{y}{8\tau^2}\right) + \dots\right). \tag{68}$$

Bearing in mind the necessity of large  $\mu\tau$  one can now apply the scaling argument (67) order by order in  $\mu$  since equation (58) and its dimensionless counterpart are linear. From (68) it is  $\alpha_0 = -1$  for the leading order,  $\alpha_0 = -2$  for the first sub-leading order and so on.

Equation (67) immediately translates to  $\langle s^n \rangle$  using (61) and (62); to leading order one finds

$$\langle s^n \rangle(\mu t; x_0) = \mu^{(3/2)n+1/2+\alpha_0} \langle s^n \rangle(t; x_0) + \dots \tag{69}$$

Assuming (1), from (2) with  $t$  taking the rôle of  $L$  it follows that  $D = 3/2$  and  $D(1 - \tau) = 1/2 + \alpha_0$ , i.e. for  $\alpha_0 = -1$  one has  $\tau = 4/3$ . The next order correction is  $D = 3/2$  and  $\tau' = 2$ .

Of course, it is also possible to calculate the leading orders of  $\langle s^n \rangle$  exactly. Because of (69), one needs to calculate  $\langle s^n \rangle(\mu t; x_0)$  for one value of  $t$  only. The simplest choice is to set  $t = x_0^2/D$ , which gives  $\langle s \rangle(x_0^2/D; x_0) = x_0^3/D$  for  $n = 1$ , i.e.

$$\langle s \rangle(t; x_0) = x_0 t \tag{70}$$

which is exactly (38) ( $n$  in (38) corresponds to  $x_0$  here and  $L$  in (38) to  $t$ ). This is actually surprising, because (70) is only the *leading* order and corrections are expected from higher orders. However, it turns out that in fact all higher order corrections cancel. Indeed, remarkably

$$\int_0^\infty dx x \exp\left(-\frac{x^2}{4t}\right) \left(\frac{x}{t} - 2 \exp\left(-\frac{1}{4t}\right) \sinh\left(\frac{x}{2t}\right)\right) = 0 \tag{71}$$

even though  $x/t$  is only the leading order of  $2 \exp(-1/(4t)) \sinh(x/(2t))$ . Especially

$$\int_0^\infty dx x \exp\left(-\frac{x^2}{4t}\right) \left(\frac{x^3}{48t^3} - \frac{x}{8t^2}\right) = 0. \tag{72}$$

According to (61) the next moment is

$$\langle s^2 \rangle(\mu t; x_0) = 2\mu^{5/2} \int_0^{tD/x_0^2} d\tau \frac{x_0^2}{D} \int_0^\infty dy \frac{x_0^3}{D} y x_0 \tilde{\psi}_1(y, \tau) \tag{73}$$

the leading order of which can be determined using the leading order of  $\tilde{\psi}_1$ ,

$$\begin{aligned} \tilde{\psi}_1(y, \tau) = \int_0^\tau d\tau' \int_0^\infty dy' \frac{1}{\sqrt{\tau'\pi}} \exp\left(-\frac{y'^2}{4\tau'}\right) \frac{y'^2}{2\tau'} \frac{1}{\sqrt{4\pi(\tau-\tau')}} \left(\exp\left(-\frac{(y-y')^2}{4(\tau-\tau')}\right) \right. \\ \left. - \exp\left(-\frac{(y+y')^2}{4(\tau-\tau')}\right)\right) + \dots \end{aligned} \tag{74}$$



which gives the leading order of  $\langle s^2 \rangle$

$$\langle s^2 \rangle(t; x_0) = \frac{32}{15\sqrt{\pi}} t^{5/2} \sqrt{Dx_0^2} + O(t^{3/2}) \quad (75)$$

identical to (55). Higher orders become very tedious, so that numerical evaluation seems to offer the better option.

## 6. Discussion and conclusion

The results above represent some of the few exact results for sandpile-like models: equations (38) and (55) are the exact leading orders of the first two moments of the avalanche size distribution without making any assumptions about scaling behaviour. The conclusion that  $\tau = 4/3$  and  $D = 3/2$  can only be drawn by either assuming (1), or by accepting the continuum result (69) and using the uniqueness of the distribution inferred from its moments<sup>2</sup>.

The method introduced in section 3 is not restricted to sandpile-like models. The underlying idea is to use a Markov matrix not only to evolve the state distribution, but also to calculate the moment generating function of the relevant observable. In order to obtain the finite-size scaling behaviour, its set of eigenvectors is generated recursively. From this recursion relation one can then develop a (discrete) PDE like (34), which can subsequently be used as a starting point for other techniques. In a two-dimensional variant of the present model, this recursion relation is much more complicated to obtain and might require the use of a matrix product state ansatz [13]. Nevertheless, it seems promising to apply the approach to more complicated processes, such as the TASEP and recent variants [28], for which there is no solution known yet.

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<sup>2</sup> Most crucially, the moment generating function  $\sum x^n \langle s^n \rangle / n!$  must actually exist, which is at least reasonable to assume from (69).

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